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HYPOELLIPTIC CONVOLUTION EQUATIONS IN K'_p, $p > 1$

BY

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ABSTRACT. We consider convolution equations in the space K'_p, $p > 1$, of distributions which "grow" no faster than $\exp(k|x|^p)$ for some constant k.

Our main result is a complete characterization of hypoelliptic convolution operators in K'_p in terms of their Fourier transforms.

In [7] and [8], the second author studied hypoelliptic convolution equations in the space S' of tempered distributions and in the space K'_1 of distributions of exponential growth. The purpose of the present paper is to extend these investigations to the space K'_p, $p > 1$, of distributions which "grow" no faster than $\exp(k|x|^p)$ for some constant k.

More precisely, we study convolution equations of the form

$$S * U = V, \tag{1}$$

where S is a distribution in $O'_c(K'_p : K'_p)$, the space of convolution operators in K'_p, and U, $V \in K'_p$. The space EK'_p of C^∞-functions in K'_p is defined in a natural way, and the equation (1) is said to be hypoelliptic in K'_p if all solutions $U \in K'_p$ are in EK'_p whenever $V \in EK'_p$.

Our main result is the following characterization of hypoelliptic convolution operators in K'_p in terms of their Fourier transforms (which are entire analytic functions).

THEOREM I. *A distribution $S \in O'_c(K'_p : K'_p)$ is hypoelliptic in K'_p if and only if its Fourier transform $\hat{S}$ satisfies the following conditions*:

(h_1) *There exist positive constants B and M such that*

$$|\hat{S}(\xi)| \geq |\xi|^{-B} \quad \text{if } \xi \in \mathbf{R}^n \text{ and } |\xi| \geq M.$$

(h_2) $|\mathrm{Im}\,\zeta|^q/\log|\zeta| \longrightarrow \infty$ *as* $|\zeta| \longrightarrow \infty$, $\zeta \in \mathbf{C}^n$, $\hat{S}(\zeta) = 0$, *where* $1/q + 1/p = 1$.

We also prove

THEOREM II. *Conditions* (h_1) *and* (h_2) *combined are equivalent to*

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(h_3) *Given $\epsilon > 0$ one can find a $B > 0$ such that for every m there exists a constant C_m so that* $|1/\hat{S}(\zeta)| \leq |\zeta|^B e^{\epsilon|\mathrm{Im}\,\zeta|^q}$ *if* $|\mathrm{Im}\,\zeta|^q \leq m \log|\zeta|$ *and* $|\zeta| \geq C_m$.

We note that, if S is in the space $\mathcal{E}'$ of distributions with compact support and $q = 1$, then conditions (h_1) and (h_2) are necessary and sufficient for S to be hypoelliptic in $\mathcal{D}'$ (see [1] and [4]).

1. **The spaces $\mathcal{K}_p$ and $\mathcal{K}'_p$.** We denote by $\mathcal{K}_p$, $p \geq 1$, the space of all functions $\varphi \in C^\infty(\mathbf{R}^n)$ such that

$$v_k(\varphi) = \sup_{x \in \mathbf{R}^n; |\alpha| \leq k} e^{k|x|^p} |D^\alpha \varphi(x)| < \infty, \qquad k = 1, 2, \ldots,$$

where $D^\alpha = (i^{-1}\partial/\partial x_1)^{\alpha_1} \cdots (i^{-1}\partial/\partial x_n)^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

The topology in $\mathcal{K}_p$ is defined by the family of seminorms v_k. Then $\mathcal{K}_p$ becomes a Frechet space and the injections $\mathcal{D} \longrightarrow \mathcal{K}_p \longrightarrow \mathcal{E}$ are continuous; here $\mathcal{E}$ denotes the space of all C^∞-functions and $\mathcal{D}$ the space of C^∞-functions with compact support (see [6]).

By $\mathcal{K}'_p$ we mean the space of continuous linear functionals on $\mathcal{K}_p$. The restriction $\widetilde{T}$ to $\mathcal{D}$ of a functional $T \in \mathcal{K}'_p$ is a distribution. Also, since $\mathcal{D}$ is dense in $\mathcal{K}_p$, T is determined by its values on $\mathcal{D}$, i.e. by $\widetilde{T}$. Thus we can identify T with $\widetilde{T}$ and regard $\mathcal{K}'_p$ as a space of distributions. We characterize the distributions in $\mathcal{K}'_p$ by their "growth" at infinity.

THEOREM 1. *A distribution $T \in \mathcal{D}'$ is in $\mathcal{K}'_p$ if and only if there exist positive integers m, k and a bounded continuous function $f(x)$ on $\mathbf{R}^n$ such that*

$$T = \frac{\partial^{mn}}{\partial x_1^m \cdots \partial x_n^m} [e^{k|x|^p} f(x)]. \tag{2}$$

PROOF. In case $p = 1$ the theorem was proved in [3]. For arbitrary $p \geq 1$, one can apply a similar argument which we present here for the sake of completeness.

It is obvious that a distribution of the form (2) defines a continuous linear functional on $\mathcal{K}_p$.

Conversely, suppose that T is in $\mathcal{K}'_p$. We first prove that, for some positive integer k_0, the set of distributions

$$\{e^{-k_0|y|^p} \tau_y T_x : y \in \mathbf{R}^n\}, \tag{3}$$

where $\tau_y T_x$ is the translation of T_x by y, is bounded in $\mathcal{D}'$.

Since T is continuous on $\mathcal{K}_p$ and the seminorms v_k are increasing, there exists $\epsilon > 0$ and a positive integer k_1 such that

$$v_k(\varphi) \leq \epsilon \quad \text{implies } |T(\varphi)| \leq 1 \tag{4}$$

for all integers $k \geq k_1$ and all $\varphi \in K_p$.

On the other hand, we have $|x-y|^p \leq 2^p(|x|^p + |y|^p)$ and therefore

$$\begin{aligned} v_k(\varphi(x+y)) &= \sup_{x\in\mathbf{R}^n;|\alpha|\leq k} e^{k|x|^p}|D^\alpha\varphi(x+y)| \\ &= \sup_{x\in\mathbf{R}^n;|\alpha|\leq k} e^{k|x-y|^p}|D^\alpha\varphi(x)| \\ &\leq e^{k2^p|y|^p} \sup_{x\in\mathbf{R}^n;|\alpha|\leq k2^p} e^{k2^p|x|^p}|D^\alpha\varphi(x)| = e^{k2^p|y|}v_{k2^p}(\varphi) \end{aligned} \tag{5}$$

for all $\varphi \in K_p$. Consequently, if $k_0 \geq k_1 2^p$, we infer from (4) and (5) that

$$|\langle e^{-k_0|y|^p}\tau_y T_x, \varphi(x)\rangle| = |\langle T_x, e^{-k_0|y|^p}\varphi(x+y)\rangle| \leq \epsilon^{-1}v_{k_0}(\varphi)$$

for all $\varphi \in \mathcal{D}$, which proves that the set (3) is bounded in $\mathcal{D}'$.

By a theorem of L. Schwartz (see [6, Vol. 2, Theorem XXII]), for every relatively compact open set $\Omega \subset \mathbf{R}^n$ there exists now an integer $N \geq 0$ and a sufficiently small compact neighborhood K of the origin such that, for every $\varphi \in \mathcal{D}_K^N$,

$$\{e^{-k_0|y|^p}\tau_y(T * \varphi): y \in \mathbf{R}^n\}$$

is a bounded set of continuous functions in Ω. It follows that $e^{-k_0|x|^p}(T*\varphi)(x)$ is a bounded, continuous function in $\mathbf{R}^n$.

Let now E be a fundamental solution for the iterated Laplace operator Δ^m, i.e. $\Delta^m E = \delta$. If m is sufficiently large, E is N times continuously differentiable and $E \in C^\infty(\mathbf{R}^n\setminus\{0\})$. Therefore, if $\gamma \in \mathcal{D}_K$ and $\gamma = 1$ in a neighborhood of the origin, we have $\gamma E \in \mathcal{D}_K^N$ and $\delta = \Delta^m(\gamma E) - W$ where $W \in \mathcal{D}_K$. Hence $T = \Delta^m(\gamma E * T) - W * T$ and so

$$T = \sum_{|\alpha|\leq m} D^\alpha[e^{k_0|x|^p}f_\alpha(x)] \tag{6}$$

where f_α are bounded, continuous functions in $\mathbf{R}^n$. Taking primitive functions, if necessary, one can reduce the right-hand side of (6) to one single term of the form (2).

We introduce in K_p' the topology of uniform convergence on all bounded sets in K_p.

2. **Convolutions in K_p'.** The convolution of a distribution $T \in K_p'$ and a function $\varphi \in K_p$ is defined as follows

$$(T * \varphi)(x) = (\varphi * T)(x) = \langle T_y, \varphi(x-y)\rangle.$$

Using Theorem 1 one can easily verify that $T * \varphi$ is a C^∞-function such that, for some integer $k_0 \geqslant 0$,

$$|D^\alpha(T * \varphi)(x)| \leqslant C_\alpha e^{k_0|x|^p},$$

where C_α are constants.

More generally, convolution operators in K_p' can be defined and characterized similarly as in K_1' by applying a method of L. Schwartz (see [5] and [3]). For simplicity we define directly the space $O_c'(K_p' : K_p')$ of distributions in K_p' which are convolution operators in K_p'.

Let γ_k, $k = 1, 2, \ldots$, be positive functions in $C^\infty(\mathbf{R}^n)$ such that

$$\gamma_k(x) = e^{k|x|^p} \quad \text{for } |x| > 1. \tag{7}$$

THEOREM 2. *For a distribution $S \in K_p'$ the following conditions are equivalent*:

(c_1) *The distributions $S_k = \gamma_k S$, $k = 1, 2, \ldots$, are in S'.*

(c_2) *For every integer $k \geqslant 0$ there exists an integer $m \geqslant 0$ such that $S = \Sigma_{|\alpha|\leqslant m} D^\alpha f_\alpha$ where f_α, $|\alpha| \leqslant m$, are continuous functions in $\mathbf{R}^n$ whose products with $e^{k|x|^p}$ are bounded.*

(c_3) *For every $\varphi \in K_p$, the convolution $S * \varphi$ is in K_p.*

PROOF. We prove the implications (c_1) $\Longleftrightarrow$ (c_2) and (c_2) $\Longleftrightarrow$ (c_3).

Suppose that condition (c_1) is satisfied and let k be an integer $\geqslant 1$. Since $S_{k+1} \in S'$, we can write $S_{k+1} = D^\alpha f$ where f is a continuous function in $\mathbf{R}^n$ such that

$$f(x) = O(1 + |x|^l) \quad \text{as } |x| \to \infty \tag{8}$$

for some integer $l \geqslant 0$. Hence $S = \gamma_{k+1}^{-1} D^\alpha f = \Sigma_{\beta\leqslant\alpha} D^{\alpha-\beta} f_\beta$ where

$$f_\beta(x) = (-1)^{|\alpha-\beta|} f(x) D^\beta \gamma_{k+1}^{-1}(x) = O(e^{-k|x|^p}) \quad \text{as } |x| \to \infty$$

in view of (7) and (8). This proves the representation (c_2).

Conversely, if (c_2) holds for some given k, then $S_k = \gamma_k S = \Sigma_{|\alpha|\leqslant m} \gamma_k D^\alpha f_\alpha$, and applying to each term of the sum the Leibnitz formula one can see that S_k is a sum of derivatives of functions which grow like polynomials. This means that S_k is in S'.

By what has just been said, the convolution $S * \varphi$ of $S \in K_p'$ and $\varphi \in K_p$

is a C^∞-function. If S satisfies condition (c_2), then

$$S * \varphi = \sum_{|\alpha| \leq m} (D^\alpha f_\alpha) * \varphi = \sum_{|\alpha| \leq m} f_\alpha * D^\alpha \varphi$$

where f_α, $|\alpha| \leq m$, are continuous functions decreasing as fast as $e^{-k|x|^p}$. Therefore

$$(9) \qquad \begin{aligned} |(f_\alpha * D^\alpha \varphi)(x)| &= \left| \int_{-\infty}^{+\infty} f_\alpha(y) D^\alpha \varphi(x-y)\, dy \right| \\ &\leq C_\alpha \int_{-\infty}^{+\infty} e^{-k\{|y|^p + |x-y|^p\}}\, dy, \end{aligned}$$

where C_α, $|\alpha| \leq m$, are constants.

Given now any integer $l \geq 0$, we choose k so large that $k > 2^p l + 1$. Since

$$(10) \qquad |x|^p \leq 2^p \{|y|^p + |x-y|^p\} \quad \text{for all } x, y \in \mathbf{R}^n,$$

we conclude from (9) that

$$|(f_\alpha * D^\alpha \varphi)(x)| \leq C_\alpha^* e^{-l|x|^p},$$

where C_α^*, $|\alpha| \leq m$, are other constants. Since l was arbitrary, we proved that $S * \varphi \in K_p$, i.e. condition (c_3) is satisfied.

Conversely, from (c_3) it follows that, for any given integer $k \geq 0$, the set of distributions

$$(11) \qquad \{e^{k|x|^p} \tau_x S_y : x \in \mathbf{R}^n\}$$

is bounded in $\mathcal{D}'$. In fact, for any $\varphi \in \mathcal{D}$, $\langle \tau_x S_y, \varphi(y)\rangle = (S * \check{\varphi})(-x)$ where $\check{\varphi}(x) = \varphi(-x)$. But $S * \check{\varphi}$ is in K_p, which shows the set (11) is bounded in $\mathcal{D}'$. Applying now an argument analogous to that used in the proof of Theorem 1 one obtains for S the representation (c_2).

We denote by $\mathcal{O}_c'(K_p' : K_p')$ the space of all distributions S satisfying the equivalent conditions (c_1)–(c_3); it is the space of convolution operators in K_p'.

If $S \in \mathcal{O}_c'(K_p' : K_p')$ and $T \in K_p'$, we define the convolution $S * T$ by

$$\langle S * T, \varphi\rangle = \langle T * S, \varphi\rangle = \langle T, \check{S} * \varphi\rangle,$$

where $\varphi \in K_p$ and $\langle \check{S}, \varphi\rangle = \langle S, \check{\varphi}\rangle$. The definition is consistent, since $\check{S} * \varphi = (S * \check{\varphi})\check{}$ is in K_p and from the proof of the implication $(c_2) \Rightarrow (c_3)$ one can see that the mapping $\varphi \longrightarrow S * \varphi$ of K_p into K_p is continuous.

If both S and T are in $\mathcal{O}_c'(K_p' : K_p')$ then $S * T$ is also in $\mathcal{O}_c'(K_p' : K_p')$. This follows from condition (c_2) and the associativity of the convolution, when all factors are in $\mathcal{O}_c'(K_p' : K_p')$.

It is also easy to prove that the convolution commutes with differentiation, i.e. $D^\alpha(S * T) = D^\alpha S * T = S * D^\alpha T$.

3. **Fourier transforms.** For a function $\varphi \in K_p$, the Fourier transform $\hat{\varphi}$ is defined by

$$\hat{\varphi}(\xi) = \int_{-\infty}^{+\infty} e^{-i\langle x, \xi\rangle} \varphi(x)\, dx$$

where $\langle x, \xi\rangle = x_1\xi_1 + \cdots + x_n\xi_n$. Also, the inversion formula holds, i.e.

$$\varphi(x) = \frac{1}{(2\pi)^n} \int_{-\infty}^{+\infty} e^{i\langle x, \xi\rangle} \hat{\varphi}(\xi)\, d\xi.$$

A distribution $S \in O'_c(K'_p : K'_p)$ is in S'; its Fourier transform $\hat{S}$ is defined by the Parseval equality

$$\langle \hat{S}, \psi\rangle = \langle S, \hat{\psi}\rangle \quad \text{for every } \psi \in S$$

(see [6, Vol. 2]).

We establish a Paley-Wiener type theorem for the spaces K_p and $O'_c(K'_p : K'_p)$. It is based on the following theorem due to G. I. Eskin [2].

THEOREM (ESKIN). *An entire analytic function $F(\zeta)$ satisfying the estimate*

$$|F(\xi + i\eta)| \leqslant C(1 + |\xi|)^N e^{A|\eta|^q} \tag{12}$$

for some constants $A, C, N > 0$ and $q > 1$, is the Fourier transform of a distribution $S \in S'$ of the form

$$S = \sum_{|\alpha| \leqslant m} D^\alpha f_\alpha \tag{13}$$

where $m = N + n + 2$ and f_α, $|\alpha| \leqslant m$, are continuous functions which, for every $\epsilon > 0$, fulfill the growth condition

$$f_\alpha(x) = O(e^{-(B-\epsilon)|x|^p}) \quad \text{as } |x| \longrightarrow \infty \tag{14}$$

with $B = p^{-1}(qA)^{-p/q}$ and $p = q/(q-1)$.

Conversely, if $S \in S'$ is of the form (13) *with the functions f_α satisfying the growth condition*

$$f_\alpha(x) = O(e^{-B|x|^p}) \quad \text{as } |x| \longrightarrow \infty, \tag{15}$$

then the Fourier transform of S is an entire analytic function $F(\zeta)$ such that

$$|F(\xi + i\eta)| \leqslant C_\epsilon(1 + |\xi|)^m e^{(A+\epsilon)|\eta|^q} \tag{16}$$

where $\epsilon > 0$ is arbitrary and C_ϵ is a constant (depending on ϵ).

Observe now that by increasing the constant B in (14) and (15) we can make A in (12) and (16) arbitrarily small. Also, for a distribution $S \in \mathcal{S}'$ we have the formula $\xi^\alpha \hat{S} = (D^\alpha S)\hat{}$.

The above observations combined with the definition of $\mathcal{K}_p$, condition (c_2) in Theorem 2 and the theorem of Eskin lead immediately to

THEOREM 3. (a) *An entire analytic function $F(\zeta)$ is a Fourier transform of a function $\varphi \in \mathcal{K}_p$ if and only if for every N and $\epsilon > 0$ there exists a constant C such that*

$$|F(\xi + i\eta)| \leq C(1 + |\xi|)^{-N} e^{\epsilon |\eta|^q}.$$

(b) *An entire analytic function $F(\zeta)$ is a Fourier transform of a distribution $S \in \mathcal{O}_c'(\mathcal{K}_p' : \mathcal{K}_p')$ if and only if for every $\epsilon > 0$ there exist constants N and C such that*

$$|F(\xi + i\eta)| \leq C(1 + |\xi|)^{N} e^{\epsilon |\eta|^q}.$$

In both (a) *and* (b), $q = p/(p-1)$, $\xi = \operatorname{Re} \zeta$ *and* $\eta = \operatorname{Im} \zeta$.

Let K_p be the space of Fourier transforms of functions in $\mathcal{K}_p$. We define in K_p a locally convex topology by means of the seminorms

$$w_k(\psi) = \sup_{\xi + i\eta \in \mathbf{C}^n} (1 + |\xi|)^k e^{-|\eta|^q/k} |\psi(\xi + i\eta)|, \qquad k = 1, 2, \ldots .$$

THEOREM 4. *The Fourier transformation is a topological isomorphism of $\mathcal{K}_p$ onto K_p.*

PROOF. By Theorem 3 and because the Fourier inversion formula is valid for functions in $\mathcal{K}_p$, the Fourier transformation is an isomorphism of $\mathcal{K}_p$ onto K_p. In view of the open mapping theorem it therefore suffices to show that the mapping $\varphi \longrightarrow \hat{\varphi}$ of $\mathcal{K}_p$ into K_p is continuous. For that purpose we observe that if k is any given integer ≥ 0 and we choose $k' \geq k^{p-1} + 1$ then, for every multi-index α with $|\alpha| \leq k$,

$$|\xi^\alpha \hat{\varphi}(\xi + i\eta)| = \left| \int_{-\infty}^{\infty} e^{-i\langle x, \xi + i\eta \rangle} D^\alpha \varphi(x)\, dx \right|$$

$$\leq \int_{-\infty}^{\infty} e^{\langle x, \eta \rangle - k'|x|^p}\, dx\, v_{k'}(\varphi) \leq \int_{-\infty}^{\infty} e^{-|x|^p}\, dx\, e^{a|\eta|^q} v_{k'}(\varphi)$$

where $a = q^{-1}\{(k'-1)p\}^{-1/(p-1)} \leq 1/k$. Hence we conclude that $w_k(\hat{\varphi}) \leq C v_{k'}(\varphi)$ for some constant C (independent of φ), which proves the desired continuity.

Let K_p' be the space of continuous linear functionals on K_p. We equip it with the topology of uniform convergence on all bounded sets in K_p. Each distribution $T \in K'_p$ has a Fourier transform $\hat{T}$ in K'_p defined by the Parseval formula $\langle \hat{T}, \hat{\varphi} \rangle = (2\pi)^n \langle T, \check{\varphi} \rangle$, $\varphi \in K_p$. Moreover, from Theorem 4 we obtain

COROLLARY. *The Fourier transformation is a topological isomorphism of K'_p onto K'_p.*

If $S \in O'_c(K'_p : K'_p)$ then, by Theorem 3, $\psi \longrightarrow \hat{S}\psi$ is a continuous linear mapping of K_p into K_p. Therefore, if $\hat{T}$ is the Fourier transform of a distribution $T \in K'_p$, one can define the product $\hat{S}\hat{T}$ by

$$\langle \hat{S}\hat{T}, \psi \rangle = \langle \hat{T}, \hat{S}\psi \rangle, \qquad \psi \in K_p.$$

Moreover, one can easily prove that $(S * T)^\wedge = \hat{S}\hat{T}$.

4. Hypoelliptic convolution equations. We denote by EK'_p the space of all C^∞-functions f such that

$$D^\alpha f(x) = O(e^{a|x|^p}) \quad \text{as } |x| \longrightarrow \infty, \tag{17}$$

for some constant a (depending on f) and all multi-indices α. Obviously EK'_p is a linear subspace of K'_p.

THEOREM 5. *If $S \in O'_c(K'_p : K'_p)$ and $f \in EK'_p$, then $S * f \in EK'_p$.*

PROOF. Suppose that f satisfies condition (17) for some a. Since S is in $O'_c(K'_p : K'_p)$, it admits the representation (c_2) in Theorem 2, i.e. for every integer $k \geqslant 0$ we can write $S = \Sigma_{|\alpha| < m} D^\alpha f_\alpha$ where f_α, $|\alpha| \leqslant m$, are continuous functions such that

$$f_\alpha(x) = O(e^{-k|x|^p}) \quad \text{as } |x| \longrightarrow \infty.$$

Choosing $k \geqslant 2^p a + 1$ and applying the inequality (10) one can see that the functions $f_\alpha(y) D^\beta f(x-y) e^{|y|^p - a|2x|^p}$ are bounded for every multi-index β. Therefore the convolutions

$$h_\alpha(x) = (f_\alpha * f)(x) = \int_{-\infty}^{\infty} f_\alpha(y) f(x-y)\, dy$$

are C^∞-functions and fulfill the growth conditions

$$D^\beta h_\alpha(x) = O(e^{a|2x|^p}) \quad \text{as } |x| \longrightarrow \infty.$$

It follows that the functions h_α are in EK'_p and consequently $S * f = \Sigma_{|\alpha| < m} D^\alpha h_\alpha$ is in EK'_p.

We now consider the convolution equation (1), i.e. $S * U = V$ where $S \in O'_c(K'_p : K'_p)$ and $U, V \in K'_p$. If there exists a solution U in EK'_p then, by Theorem 5, V must be in EK'_p.

Conversely, if all solutions $U \in K'_p$ are in EK'_p whenever V is in EK'_p, the equation (and the distribution S) is said to be hypoelliptic in $\mathcal{K}_p'$. In the next two sections we prove that conditions (h_1) and (h_2) in Theorem I are necessary and sufficient for S to be hypoelliptic in $\mathcal{K}_p'$.

5. **Necessity of conditions (h_1) and (h_2).** In order to prove the necessity of condition (h_1) we first study series of the form

$$\sum_{j=1}^{\infty} a_j \delta_{({}_j\xi)} \tag{18}$$

where $\delta_{({}_j\xi)}$ is the δ-Dirac measure with singularity at ${}_j\xi \in \mathbf{R}^n$ and the coefficients a_j are complex numbers. We assume that

$$|{}_j\xi| > 2|{}_{j-1}\xi| > 2^j, \qquad j = 1, 2, \ldots, \tag{19}$$

and

$$a_j = O(|{}_j\xi|^{\mu}) \quad \text{as } j \longrightarrow \infty, \tag{20}$$

for some integer $\mu > 0$. Then the series (18) converges in S'.

The following lemma is a slightly strengthened version of Lemma 1 in [7].

LEMMA 1. *Suppose that T is a distribution in S' whose Fourier transform $\hat{T}$ is of the form* (18), *i.e.*

$$\hat{T} = \sum_{j=1}^{\infty} a_j \delta_{({}_j\xi)} \tag{21}$$

where ${}_j\xi$ and a_j, $j = 1, 2, \ldots$, satisfy conditions (19) *and* (20).

If, for every integer $\nu > 0$,

$$a_j = O(|{}_j\xi|^{-\nu}) \quad \text{as } j \longrightarrow \infty, \tag{22}$$

then T is a C^∞-function bounded in $\mathbf{R}^n$ together with all its derivatives.

Conversely, if condition (22) *is not fulfilled, then T is not in $C^\infty(\mathbf{R}^n)$.*

PROOF. By virtue of (20) and (21), $T = (2\pi)^{-n}\sum_{j=1}^{\infty} a_j e^{i\langle x, {}_j\xi\rangle}$ where the series converges in S'. If the coefficients a_j satisfy condition (22), the last series converges uniformly in $\mathbf{R}^n$ together with all its term-by-term derivatives, which proves the first part of the lemma.

Conversely, if $T \in C^\infty(\mathbf{R}^n)$ then, for every $\nu \geqslant 0$ and $\varphi \in \mathcal{D}$,

$$\langle e^{-i\langle h,x\rangle} \Delta^\nu T_x, \varphi(-x)\rangle \longrightarrow 0 \quad \text{as } |h| \longrightarrow \infty,$$

$h \in \mathbf{R}^n$. Hence

$$\langle \tau_{-h}(|\xi|^{2\nu}\hat{T}_\xi), \hat{\varphi}(\xi)\rangle = \sum_{j=1}^{\infty} a_j |{}_j\xi|^{2\nu} \hat{\varphi}({}_j\xi - h) \longrightarrow 0, \tag{23}$$

by application of the Parseval identity.

We now choose φ so that

$$|\hat{\varphi}(0)| \geq 1. \tag{24}$$

If condition (22) is not satisfied, then there exists $\rho > 0$ and an integer $\nu_0 > 0$ such that

$$|_j\xi|^{2\nu_0}|a_j| \geq \rho \tag{25}$$

for a subsequence of $\{a_j\}$, but without loss of generality we take the whole sequence. Also, since $\hat{\varphi} \in S$, we have

$$\hat{\varphi}(\xi) = O(|\xi|^{-\mu-2\nu_0-1}) \quad \text{as } |\xi| \longrightarrow \infty,\ \xi \in \mathbf{R}^n. \tag{26}$$

Setting $_jh = {}_j\xi$ and making use of (19), (20) and (26) we obtain

$$\sum_{j=1;j\neq k}^{\infty} a_j|_j\xi|^{2\nu_0}\hat{\varphi}(_j\xi - {}_kh) = O(2^{-k}) \quad \text{as } k \longrightarrow \infty.$$

On the other hand, conditions (24) and (25) imply that $|a_k|\,|_k\xi|^{2\nu_0}|\hat{\varphi}(0)| \geq \rho$. This contradicts the convergence (23). Our assertion is thus proved.

THEOREM 6. *If a distribution $S \in O_c'(K_p' : K_p')$ is hypoelliptic in K_p' then its Fourier transform $\hat{S}$ satisfies condition* (h_1).

PROOF. If condition (h_1) is not satisfied then there exists a sequence $\{_j\xi\} \subset \mathbf{R}^n$ defined as in Lemma 1, such that

$$|\hat{S}(_j\xi)| < |_j\xi|^{-j}. \tag{27}$$

The distribution $\hat{U} = \sum_{j=1}^{\infty}\delta_{(_j\xi)}$ is in S' and, by Lemma 1, U is not in $C^\infty(\mathbf{R}^n)$. But

$$(S * U)\hat{} = \hat{S}\hat{U} = \sum_{j=1}^{\infty} \hat{S}(_j\xi)\delta_{(_j\xi)}$$

whence, applying (27) and again Lemma 1, we conclude that $S * U$ is in EK_p'. Thus S is not hypoelliptic in K_p'.

REMARK. One can modify Lemma 1 and Theorem 6 so as to obtain a proof of the necessity of condition (h_3).

In the proof of condition (h_2) we may restrict ourselves to continuous solutions of the homogeneous equation.

THEOREM 7. *If every continuous solution of the homogeneous equation*

$$S * U = 0 \tag{28}$$

which satisfies the estimate

(29) $$U(x) = O(e^{k|x|^p}) \quad as\ |x| \longrightarrow \infty$$

for some $k > 0$, has continuous first order derivatives in a neighborhood of the origin, then condition (h_2) *holds.*

PROOF. Without restriction of generality we may assume that the neighborhood of the origin is the ball $B = \{x\colon |x| \leqslant 1\}$.

Fix now $k > 0$ and denote by $H_{p,k}$ the space of continuous solutions of equation (28) satisfying the estimate (29). In $H_{p,k}$ we define the norm

$$\|U\| = \sup_{x \in \mathbf{R}^n} |U(x)| e^{-k|x|^p}.$$

We also denote by $H^*_{p,k}$ the subspace of $H_{p,k}$ consisting of functions with continuous first order derivatives in B; a norm in $H^*_{p,k}$ is defined by

$$\|U\|^* = \|U\| + \sup_{|x|<1} \sum_{j=1}^{n} \left| \frac{\partial u}{\partial x_j}(x) \right|.$$

It is easy to see that $H_{p,k}$ and $H^*_{p,k}$ are Banach spaces. By assumption $H_{p,k}$ is mapped by the identity mapping onto $H^*_{p,k}$. The mapping is closed and therefore continuous. Thus there exists a constant $C > 0$ such that

(30) $$\|U\|^* \leqslant C\|U\| \quad \text{for all } U \in H_{p,k}.$$

If $\hat{S}(\zeta) = 0$ for some $\zeta \in \mathbf{C}^n$, then $U_0(x) = e^{i\langle \zeta, x\rangle}$ is a solution of equation (28) and $U_0 \in H_{p,k}$. But

$$\|U_0\| = \sup_{x \in \mathbf{R}^n} e^{-\langle \eta, x\rangle - k|x|^p} = e^{B|\eta|^q}$$

where $B = q^{-1}(kp)^{-q/p}$, and $\|U_0\|^* = \|U_0\| + \Sigma_{j=1}^n |\zeta_j| e^{|\eta|} \geqslant \|U_0\| + |\zeta|$. Thus, from (30) we obtain

$$\lim_{|\zeta| \to \infty} \frac{|\eta|^q}{\log|\zeta|} \geqslant q(kp)^{q/p},$$

and since k is arbitrary we conclude that condition (h_2) is valid.

COROLLARY. *If $S \in \mathcal{O}'_c(K'_p : K'_p)$ is hypoelliptic in K'_p, then condition* (h_2) *holds.*

6. **The implication (h_1), (h_2) ⇒ (h_3).** For $q = 1$ this implication was proved by L. Hörmander [4]. In this section, we modify suitably Hörmander's argument to our case. First we establish the following lemma on harmonic functions in $\mathbf{R}^2$.

LEMMA 2. *Given $A, B, b > 0$ and $q > 1$ one can find a constant $N > 0$*

such that if u is a harmonic function for $x^2 + y^2 < \rho^2$ *and satisfies the inequalities*

$$(31)\qquad u(x, 0) \leqslant 0, \quad u(x, y) \geqslant -a|y|^q - B\rho^q, \quad x^2 + y^2 < \rho^2,$$

then it follows that

$$(32)\qquad u(x, y) \leqslant a|y|^q + (B + b)r^q, \quad x^2 + y^2 < r^2$$

provided that $0 < a < A$ *and* $0 < r \leqslant \rho/N$.

PROOF. Assuming that the lemma is false one can find a harmonic function v in $\mathbf{R}^2$ and real numbers a_0, x_0, y_0 such that

$$(33)\quad v(x, 0) \leqslant 0, \quad v(x, y) \geqslant -a_0|y|^q - B, \quad v(x_0, y_0) \geqslant a_0|y_0|^q + (B + b).$$

This can be accomplished in the same way as in the proof of Lemma 1 in [4].

From the first two inequalities in (33) it follows that v is a linear function of y, since v must be a harmonic polynomial which is bounded when y is bounded. Suppose that $v(x, y) = cy + d$. Then $d \leqslant 0$, by the first inequality in (33), and from the remaining two inequalities in (29) it follows that

$$(34)\qquad a_0|y|^q + cy + B + d \geqslant 0$$

and

$$(35)\qquad a_0|y_0|^q - cy_0 + (B + b) - d \leqslant 0.$$

In particular, setting $y = -y_0$ in (34) we obtain $a_0|y_0|^q - cy_0 + B + d \geqslant 0$ which contradicts the inequality (35), because $d \leqslant 0$.

Let now S be a distribution in $O'_c(K'_p : K'_p)$ whose Fourier transform satisfies conditions (h_1) and (h_2). Then there exist constants $B_1, M_1 > 0$ such that

$$(36)\qquad |\hat{S}(\xi)| \geqslant |\xi|^{-B_1} \quad \text{if } \xi \in \mathbf{R}^n, |\xi| \geqslant M_1,$$

and

$$(37)\qquad |\eta|^q/\log|\zeta| \longrightarrow \infty \quad \text{if } \zeta = \xi + i\eta \in \mathbf{C}^n, |\zeta| \longrightarrow \infty, \hat{S}(\zeta) = 0.$$

Furthermore, by Theorem 3, to every $\epsilon > 0$ there exist constants $B_2, M_2 > 0$ so that

$$(38)\qquad |\hat{S}(\zeta)| \leqslant |\xi|^{B_2} e^{\epsilon|\eta|^q} \quad \text{if } \zeta \in \mathbf{C}^n, |\zeta| \geqslant M_2.$$

THEOREM 8. *If the Fourier transform* $\hat{S}$ *of a distribution* $S \in O'_c(K'_p : K'_p)$ *satisfies conditions* (36), (37) *and* (38), *then to every* $m > 0$ *one can find a* *constant* C_m *such that*

(39) $$|1/\hat{S}(\zeta)| \leqslant |\xi|^{2(B_1+B_2+1)} e^{\epsilon|\eta|^q} \quad \text{if } |\eta|^q \leqslant m\log|\zeta|$$

and $|\zeta| \geqslant C_m$.

PROOF. Let M be a positive constant which we fix later. Given $\zeta \in \mathbf{C}^n$ such that $0 < |\eta|^q < m\log|\zeta|$ we consider the analytic function of one complex variable z defined by

$$F_\zeta(z) = \hat{S}(\xi + z\eta/|\eta|) \quad \text{for } |z|^q \leqslant M\log|\xi|.$$

If $|\zeta|$ is sufficiently large, conditions (36), (37) and (38) imply that

(40) $$|F_\zeta(x)| \geqslant (2|\xi|)^{-B_1} \quad \text{if } x \in \mathbf{R},\ |x|^q < M\log|\xi|,$$

and

(41) $$|F_\zeta(z)| \leqslant (2|\xi|)^{B_2} e^{\epsilon|y|^q} \quad \text{if } |z|^q < M\log|\xi|.$$

The function $u_\zeta(z) = \log\{(2|\xi|)^{-B_1}|F_\zeta(z)|^{-1}\}$ is harmonic for $|z|^q < M\log(\xi)$ and large $|\zeta|$, by virtue of (37). Moreover, from (40) and (41) it follows that

(42) $$u_\zeta(x) \leqslant 0 \quad \text{if } x \in \mathbf{R} \text{ and } |x|^q < M\log|\xi|$$

and

(43) $$u_\zeta(z) \geqslant -\epsilon|y|^q - (B_1 + B_2)\log(2|\xi|), \qquad |z|^q < M\log|\xi|.$$

We now apply Lemma 2 with the constants $A = 1 + \epsilon$, $B = (B_1 + B_2 + 1)/(m + 1)$, $b = 1/(m + 1)$ and $r^q = (m + 1)\log|\xi|$. If N is the constant in the lemma, we set $M = N^q(m + 1)$ and we observe that

$$(B_1 + B_2)\log(2|\xi|) \leqslant (B_1 + B_2 + 1)\log|\xi| = Br^q,$$

if $|\zeta|$ is sufficiently large. Thus, by Lemma 2, we have

(44) $$u_\zeta(z) \leqslant \epsilon|y|^q + (B_1 + B_2 + 2)\log|\xi| \quad \text{if } |z|^q \leqslant r^q = (m + 1)\log|\xi|.$$

Since

$$|\eta|^q \leqslant m\log|\zeta| \leqslant (m + 1)\log|\xi| = r^q$$

if $|\zeta|$ is sufficiently large, we may substitute $z = i|\eta|$ in (44). Then we obtain

$$\log\{(2|\xi|)^{-B_1}|\hat{S}(\zeta)|^{-1}\} \leqslant \epsilon|\eta|^q + (B_1 + B_2 + 2)\log|\xi|$$

whence we conclude that

$$|1/\hat{S}(\zeta)| \leqslant |\xi|^{2B_1+B_2+2}e^{\epsilon|\eta|^q} \leqslant |\xi|^{2(B_1+B_2+1)}e^{\epsilon|\eta|^q}$$

where $|\zeta|$ is sufficiently large. This proves Theorem 8.

COROLLARY. *For a distribution* $S \in O'_c(K'_p : K'_p)$, *conditions* (h_1) *and* (h_2) *combined imply condition* (h_3).

7. **Parametrices.** We define suitable parametrices for a distribution $S \in O'_c(K'_p : K'_p)$ and prove that these parametrices exist if S fulfills condition (h_3).

In what follows b is a positive number and k a positive integer.

DEFINITION. A distribution $P \in K'_p$ is said to be a (b, k)-parametrix for S if it has the following properties:

(p_1) There exists an integer $m > 0$ such that $P = \Sigma_{|\alpha|<m} D^\alpha F_\alpha$ where F_α, $|\alpha| \leqslant m$, are continuous functions in $\mathbf{R}^n$ such that $F_\alpha(x) = O(e^{-b|x|^p})$ as $|x| \longrightarrow \infty$,

(p_2) $S * P = \delta - W$, where δ is the Dirac measure and W is a function in $C^k(\mathbf{R}^n)$ satisfying the growth condition $D^\alpha W(x) = O(e^{-b|x|^p})$ as $|x| \longrightarrow \infty$ when $|\alpha| \leqslant k$.

THEOREM 9. *Let S be a distribution in $O'_c(K'_p : K'_p)$ which satisfies condition* (h_3). *Then for each pair (b, k) there exists a (b, k)-parametrix for S.*

PROOF. In order to simplify the notation we present the proof of Theorem 9 for $n = 2$. The general case can be handled in the same way.

We apply condition (h_3) with ϵ and m to be fixed later. Suppose that (h_3) holds for some given ϵ, m, $B > 0$ and $C_m > 1$. Then the function

$$F(x, \zeta) = \{(2\pi)^2 \hat{S}(\zeta)\langle\zeta, \zeta\rangle^\mu\}^{-1} e^{i\langle x, \zeta\rangle}$$

is analytic in ζ, when $|\eta|^q \leqslant m \log|\zeta|$ and $|\zeta| \geqslant C_m$, provided that C_m is sufficiently large. If $\mu > B/2 + 1$, then $F(x, \xi)$ is integrable over $R^2 \backslash Q$, where $Q = \{\xi \in R^2 \colon |\xi_j| \leqslant C_m, j = 1, 2\}$. Moreover, if μ is even and

$$h(x) = \iint_{R^2 \backslash Q} F(x, \xi)\, d\xi_2\, d\xi_1, \tag{45}$$

then it is easy to verify that the distribution

$$H = \Delta^\mu h \tag{46}$$

satisfies the equation

$$S * H = \delta - \frac{1}{(2\pi)^2} \iint_Q e^{i\langle x, \xi\rangle}\, d\xi_2\, d\xi_1. \tag{47}$$ [1]

The integral in (45) can be represented as

---

[1] In what follows we assume that μ is even; otherwise we should need $(-\Delta)^\mu$ in (46).

$$(48)\qquad \iint_{R^2\setminus Q} = \int_{|\xi_1|\geq C_m}\int_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\int_{|\xi_2|\geq C_m} - \int_{|\xi_1|\geq C_m}\int_{|\xi_2|\geq C_m}.$$

We now replace the iterated integrals in (48) by integrals over contours in the respective complex planes.

Let σ be a C^∞-function of one real variable r defined for $r > 0$ in such a way that $\sigma(r) = C_m$ for $0 < r \leqslant C_m$, σ is increasing for $r \geqslant C_m$, and $\sigma(r) = C_m e^{ar^p}$ for $r \geqslant 2C_m$, where a is a positive constant. We extend σ to negative values of r by setting $\sigma(-r) = -\sigma(r)$. We also denote by σ^* an increasing, odd C^∞-function on R such that $d^k\sigma/dx^k|_{x=0} = 0$, $k = 1, 2, \ldots$, and $\sigma^*(r) = \sigma(r)$ for $r \geqslant C_m$.

Further, let τ be an even, C^∞-function on R such that $\tau(r) = 0$ for $|r| \leqslant C_m$, τ is increasing for $r \geqslant C_m$, and $\tau(r) = c|r|^{p/q}$ for $|r| \geqslant 2C_m$, where c is a positive constant.

We assume that

$$(49)\qquad (\sqrt{2}\,\tau(r))^q \leqslant m \log|\sigma(r)| \quad \text{for} \quad |r| \geqslant C_m$$

which implies that

$$(50)\qquad (\sqrt{2}\,c)^q \leqslant ma.$$

Given any $x = (x_1, x_2)$ in $\mathbf{R}^2$ we denote by Λ_j (or Λ_j^*, respectively) the contour in the complex ζ_j plane defined by $\zeta_j(r) = \sigma(r) + i \operatorname{sgn} x_j\tau(r)$, where r runs from $-\infty$ to $-C_m$ and from C_m to ∞ (or by $\zeta_j(r) = \sigma^*(r) + i \operatorname{sgn} x_j\tau(r)$, where r runs from $-\infty$ to ∞, respectively). By the inequality (49), the contours $\Lambda_1 \times \Lambda_2^*$, $\Lambda_1^* \times \Lambda_2$ and $\Lambda_1 \times \Lambda_2$ lie in the domain $|\eta|^q < m\log|\zeta|$. If, in addition,

$$(51)\qquad \mu > B + \epsilon m + 1$$

then we can write

$$(52)\qquad h(x) = \left(\int_{\Lambda_1}\int_{\Lambda_2^*} + \int_{\Lambda_1^*}\int_{\Lambda_2} - \int_{\Lambda_1}\int_{\Lambda_2}\right) F(x, \zeta)\, d\zeta_2\, d\zeta_1.$$

We denote by λ_j and λ_j^* the parts of the contours Λ_j and Λ_j^*, respectively, obtained by restricting the values of the parameter r to the interval $(-|x|, |x|)$. The remaining portions of Λ_j and Λ_j^* will be denoted by Γ_j and Γ_j^*, respectively.

If

$$h_1(x) = \left(\int_{\Gamma_1}\int_{\Gamma_2^*} + \int_{\Gamma_1^*}\int_{\Gamma_2} - \int_{\Gamma_1}\int_{\Gamma_2}\right) F(x, \zeta)\, d\zeta_2\, d\zeta_1,$$

we now set $P = \Delta^\mu h_1$. Then, by virtue of (46), (47) and (52), we have

$$S * P = \delta - W \tag{53}$$

where

$$W = S * \Delta^\mu h_2 + \frac{1}{(2\pi)^2} \iint_Q e^{i\langle x,\xi\rangle} \, d\xi_2 \, d\xi_1$$

and

$$h_2(x) = \left(\int_{\lambda_1}\int_{\lambda_2^*} + \int_{\lambda_1^*}\int_{\lambda_2} - \int_{\lambda_1}\int_{\lambda_2}\right) F(x, \zeta)\, d\zeta_2\, d\zeta_1.$$

The proof of the theorem follows immediately from the next two technical lemmas.

LEMMA 3. *The function h_1 satisfies the growth condition*

$$h_1(x) = O(e^{-c|x|^p}) \quad \text{as } |x| \longrightarrow \infty. \tag{54}$$

PROOF. Consider first the integral

$$\int_{\Gamma_1}\int_{\Gamma_2} F(x, \zeta)\, d\zeta_2\, d\zeta_1 = \int_{|t_1| \geq |x|}\int_{|t_2| \geq |x|} F(x, \zeta(t))\zeta_1'(t_1)\zeta_2'(t_2)\, dt_2\, dt_1$$

where $\zeta(t) = (\zeta_1(t_1), \zeta_2(t_2))$ and $\zeta_j(t_j) = \sigma(t_j) + i \operatorname{sgn} x_j \tau(t_j), j = 1, 2$.

For $|t_1|$ and $|t_2|$ sufficiently large, we have

$$|\langle \zeta(t), \zeta(t)\rangle|^\mu \geq \exp\{\mu a(|t_1|^p + |t_2|^p)\}$$

and

$$|\zeta_1'(t_1)\zeta_2'(t_2)| \leq \tfrac{1}{2}|t|^p \exp\{a(|t_1|^p + |t_2|^p)\}.$$

Also, from (h_3) it follows that

$$\left|\frac{1}{\hat{S}(\zeta(t))}\right| \leq |\zeta(t)|^B \exp\{\epsilon|\eta(t)|^q\}$$

$$\leq 2\, C_m^B \exp\{(Ba + \epsilon(c\sqrt{2})^q)(|t_1|^p + |t_2|^p)\},$$

provided that $|t_1|$ and $|t_2|$ are sufficiently large.

Further, if $|t_1|, |t_2| \geq |x|$, we have

$$|\exp\{i\langle x, \zeta(t)\rangle\}| = \exp\{-c(|x_1|\,|t_1|^{p/q} + |x_2|\,|t_2|^{p/q})\}$$

$$\leq \exp\{-c(|x_1| + |x_2|)|x|^{p/q}\} \leq \exp\{-c|x|^p\}.$$

Consequently, for $|t_1|$ and $|t_2|$ sufficiently large and $\geq |x|$,

$$
\begin{aligned}
&|F(x, \zeta(t))\zeta_1'(t_1)\zeta_2'(t_2)| \\
&\qquad \leq (2\pi)^{-2} C_m^B \exp\{-c|x|^p\}|t|^p \\
&\qquad\quad \times \exp\{[Ba + a - \mu a + \epsilon(c\sqrt{2})^q](|t_1|^p + |t_2|^p)\}.
\end{aligned} \tag{55}
$$

But

$$Ba + a - \mu a + \epsilon(c\sqrt{2})^q \leq a(B + 1 - \mu + \epsilon m) < 0,$$

because of (50) and (51). Therefore the right-hand side of (55) is integrable with respect to t_1 and t_2 over $|t_1|, |t_2| \geq C_m$, which proves the growth condition (54) for the integral under consideration.

We now observe that

$$\int_{\Gamma_1^*} \int_{\Gamma_2} F(x, \zeta)\, d\zeta_2\, d\zeta_1 = \int_{l_1} \int_{\Gamma_2} F(x, \zeta)\, d\zeta_2\, d\zeta_1 \tag{56}$$

where l_1 is the line $\zeta_1(t_1) = t_1 + i \operatorname{sgn} x_1 \tau(|x|)$ directed from $-\infty$ to ∞. The last integral is equal to

$$\int_{-\infty}^{\infty} \int_{|t_2| \geq |x|} F(x, \zeta(t))\zeta_2'(t_2)\, dt_2\, dt_1,$$

where $\zeta(t) = (\zeta_1(t_1), \zeta_2(t_2))$, $\zeta_1(t_1) = t_1 + i \operatorname{sgn} x_1 \tau(|x|)$ and $\zeta_2(t_2) = \sigma(t_2) + i \operatorname{sgn} x_2 \tau(t_2)$.

If $|t_1|, |t_2|$ are sufficiently large and $|t_2| \geq |x|$, we now have

$$|\langle \zeta(t), \zeta(t)\rangle|^\mu \geq \tfrac{1}{2}(t_1^2 + e^{2a|t_2|^p})^\mu,$$

$$|\zeta_2'(t_2)| \leq \tfrac{1}{2}|t_2|^p e^{a|t_2|^p},$$

and, in view of (h_3) and (49),

$$
\begin{aligned}
|1/\hat{S}(\zeta(t))| &\leq (t_1^2 + e^{2a|t_2|^p})^B e^{\epsilon(c\sqrt{2})^q |t_2|^p} \\
&\leq (t_1^2 + e^{2a|t_2|^p})^{B + m\epsilon/2}.
\end{aligned}
$$

Also, for $|t_2| \geq |x|$,

$$
\begin{aligned}
|\exp\{i\langle x, \zeta(t)\rangle\}| &\leq \exp\{-c(|x_1|\,|x|^{p/q} + |x_2|\,|t_2|^{p/q})\} \\
&\leq \exp\{-c(|x_1| + |x_2|)|x|^{p/q}\} \leq \exp\{-c|x|^p\},
\end{aligned}
$$

so that, finally,

$$
\begin{aligned}
|F(x, \zeta(t))\zeta_2'(t_2)| &\leq (2\pi)^{-2} e^{-c|x|^p} |t_2|^p e^{a|t_2|^p} (t_1^2 + e^{2a|t_2|^p})^{B - \mu + m\epsilon/2} \\
&\leq (2\pi)^{-2} e^{-c|x|^p} |t_2|^p e^{a|t_2|^p} (t_1^2 + e^{2a|t_2|^p})^{-1 - m\epsilon/2}
\end{aligned} \tag{57}
$$

because of (51). Since the last expression in (57) is integrable with respect to t_1 and t_2 for $t_1 \in \mathbf{R}$ and $|t_2| \geqslant C_m$, condition (54) holds for the integral (56). The integral $\int_{\Gamma_1}\int_{\Gamma_2^*} F(x, \zeta)\, d\zeta_2\, d\zeta_1$ can be estimated in the same way. Lemma 3 is now established.

LEMMA 4. *Given any pair* (b, k) *we can choose the constants* ϵ, a *(sufficiently small) and* m, c *(sufficiently large) so that*

$$D^\alpha W(x) = O(e^{-b|x|^p}) \quad \text{as } |x| \longrightarrow \infty$$

for $|\alpha| \leqslant k$.

PROOF. Assume that $|x| \longrightarrow \infty$ through $x_1, x_2 \geqslant 0$; otherwise we could modify suitably our argument.

By definition,

$$D^\alpha W = S * D^\alpha \Delta^\mu h_2 + \frac{1}{(2\pi)^2} D^\alpha \iint_Q e^{i\langle x, \xi\rangle}\, d\xi_2\, d\xi_1 \tag{58}$$

where

$$h_2(x) = \left(\int_{\Lambda_1}\int_{\Lambda_2^*} + \int_{\Lambda_1^*}\int_{\Lambda_2} - \int_{\Lambda_1}\int_{\Lambda_2}\right) F(x, \zeta)\, d\zeta_2\, d\zeta_1.$$

It is easy to verify that h_2 is a C^∞-function such that $h_2(x) = 0$ for $|x| \leqslant C_m$ and

$$D^\alpha h_2(x) = O(e^{a(|\alpha|+1)|x|^p}) \quad \text{as } |x| \longrightarrow \infty$$

for all α.

On the other hand, by Theorem 2, for every integer ρ one can find an integer l such that $S = \Sigma_{|\beta|<l} D^\beta f_\beta$, where f_β, $|\beta| \leqslant l$, are continuous functions in $\mathbf{R}^n$ satisfying the condition

$$f_\beta(x) = O(e^{-\rho|x|^p}) \quad \text{as } |x| \longrightarrow \infty. \tag{59}$$

Therefore, if a is so small that $(2\mu + k + l + 1)a < \rho$, we can write

$$S * D^\alpha \Delta^\mu h_2 = \sum_{|\beta|<l} (-1)^{|\alpha+\beta|} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_\beta(y) D_y^{\alpha+\beta} \Delta_y^\mu h_2(x - y)\, dy_2\, dy_1 \tag{60}$$

where $|\alpha| \leqslant k$.

We decompose $h_2(x - y)$ as follows: $h_2(x - y) = g_1(x, y) + g_2(x, y)$ where $g_1(x, y)$ is defined similarly as $h_2(x - y)$ except for the contours which are parts of $\Lambda_1, \Lambda_2, \Lambda_1^*, \Lambda_2^*$ obtained by restricting the parameters t_1, t_2 to the interval $[-|x|, |x|]$ (instead of $[-|x - y|, |x - y|]$).

The function $g_1(x, y)$ is a sum of three integrals one of which is

$$(61)\qquad \int_{-|x|}^{|x|}\int_{|x|\geq|t_2|\geq C_m} F(x-y,\zeta(t))\zeta_1'(t_1)\zeta_2'(t_2)\,dt_2\,dt_1$$

where $\zeta(t) = (\zeta_1(t_1), \zeta_2(t_2))$, $\zeta_1(t_1) = \sigma^*(t_1) + \operatorname{sgn}(x_1 - y_1)\tau(t_1)$ and $\zeta_2(t_2) = \sigma(t_2) + \operatorname{sgn}(x_2 - y_2)\tau(t_2)$. Its contribution toward the right-hand side of (60) is

$$\frac{1}{(2\pi)^2}D^\alpha\int_{\lambda_1^*}\int_{\lambda_2} e^{i\langle x,\zeta\rangle}\,d\zeta_2\,d\zeta_1$$

$$+\sum_{|\beta|<l}(-1)^{|\alpha+\beta|}\left\{\int_{-\infty}^{x_1}\int_{x_2}^{\infty}\left[f_\beta(y)D_y^{\alpha+\beta}\Delta_y^\mu\int_{-|x|}^{|x|}\int_{-\tau(|x|)}^{\tau(|x|)}F_1(x,y,t)\,dt_2\,dt_1\right]dy_2\,dy_1\right.$$

$$(62)\qquad +\int_{x_1}^{\infty}\int_{-\infty}^{x_2}\left[f_\beta(y)D_y^{\alpha+\beta}\Delta_y^\mu\int_{-\tau(|x|)}^{\tau(|x|)}\int_{|x|\geq|t_2|\geq C_m}F_2(x,y,t)\,dt_2\,dt_1\right]dy_2\,dy_1$$

$$\left.+\int_{x_1}^{\infty}\int_{x_2}^{\infty}\left[f_\beta(y)D_y^{\alpha+\beta}\Delta_y^\mu\int_{-\tau(|x|)}^{\tau(|x|)}\int_{-\tau(|x|)}^{\tau(|x|)}F_3(x,y,t)\,dt_2\,dt_1\right]dy_2\,dy_1\right\}$$

where the functions F_1, F_2, F_3 are defined by

$$F_1(x,y,t) = \sum_{j=0}^{1}(-1)^jF(x-y,{}_j\zeta(t))_j\zeta_1'(t_1)_j\zeta_2'(t_2),$$

$$F_2(x,y,t) = \sum_{j=2}^{3}(-1)^jF(x-y,{}_j\zeta(t))_j\zeta_1'(t_1)_j\zeta_2'(t_2),$$

$$F_3(x,y,t) = \sum_{r,s=0}^{1}(-1)^{r+s}F(x-y,{}_{r,s}\zeta(t))_{r,s}\zeta_1'(t_1)_{r,s}\zeta_2'(t_2),$$

where

$$_j\zeta(t) = ({}_j\zeta_1(t_1), {}_j\zeta_2(t_2)), \qquad {}_{r,s}\zeta(t) = ({}_{r,s}\zeta_1(t_1), {}_{r,s}\zeta_2(t_2))$$

$$_j\zeta_1(t_1) = \sigma^*(t_1) + i\tau(|x|), \qquad {}_j\zeta_2(t_2) = (-1)^j\sigma(|x|) + it_2, \quad j = 0, 1,$$

$$_j\zeta_1(t_1) = (-1)^j\sigma^*(|x|) + it_1, \qquad {}_j\zeta_2(t_2) = \sigma(t_2) + i\tau(|x|), \quad j = 2, 3,$$

$$_{r,s}\zeta_1(t_1) = (-1)^r\sigma^*(|x|) + it_1, \qquad {}_{r,s}\zeta_2(t_2) = (-1)^s\sigma(|x|) + it_2.$$

In deriving (62) we made repeated use of Cauchy's integral theorem.

For sufficiently large $|x|$ each of the integrals in curly brackets in (62) is less than

$$(63)\quad \exp\{\{-c + c^q[\epsilon 2^{q/2} + (2/q)(1/p\rho)^{1/(p-1)}] + a(B + k + l + 1)\}|x|^p\}.$$

Given $b > 0$, we now fix ϵ so small and ρ (or equivalently l) so large that

$$\epsilon 2^{q/2} + (2/q)(1/p\rho) < 1/q(2bp)^{q-1}.$$

Next we choose $c = 2bp$ and $a = b/(B + k + l + 1)$. Then the expression in (63) is less than $e^{-b|x|^p}$. In order to satisfy condition (50) we have, however, to choose $m \geq a^{-1}(c\sqrt{2})^q$.

A representation similar to (62) can be obtained for the contribution toward the right-hand side of (60) of the two remaining integrals which make up $g_1(x, y)$. The sum of the first terms in these representations is

$$\frac{1}{(2\pi)^2} D^\alpha \left(\int_{\lambda_1^*} \int_{\lambda_2} + \int_{\lambda_1} \int_{\lambda_2^*} - \int_{\lambda_1} \int_{\lambda_2} \right) e^{i\langle x, \zeta\rangle} \, d\zeta_2 \, d\zeta_1$$

$$= \frac{1}{(2\pi)^2} D^\alpha \int_{\lambda_1^*} \int_{\lambda_2^*} e^{i\langle x, \zeta\rangle} \, d\zeta_2 \, d\zeta_1 - \frac{1}{(2\pi)^2} D^\alpha \iint_Q e^{i\langle x, \zeta\rangle} \, d\zeta_2 \, d\zeta_1$$

and, for sufficiently large $|x|$, we have

$$\left| D^\alpha \int_{\lambda_1^*} \int_{\lambda_2^*} e^{i\langle x, \zeta\rangle} \, d\zeta_2 \, d\zeta_1 \right| \leqslant e^{[-c+a(k+2)]\,|x|^p} \leqslant e^{-b|x|^p},$$

under the above choice of c and a.

In view of (58) the proof of the lemma will be complete if we can choose ϵ, a sufficiently small and c, m sufficiently large to insure that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_\beta(y) D_y^{\alpha+\beta} \Delta_y^\mu g_2(x, y) \, dy_2 \, dy_1 = O(e^{-b|x|^p}) \tag{64}$$

as $|x| \longrightarrow \infty$, for $|\alpha| \leqslant k$ and $|\beta| \leqslant l$. We again prove (64) for one of the three integrals of which $g_2(x, y)$ is composed, namely for

$$\int_{-|x-y|}^{|x-y|} \int_{|x-y| \geqslant |t_2| \geqslant |x|} F(x-y, \zeta(t)) \zeta_1'(t_1) \zeta_2'(t_2) \, dt_2 \, dt_1 \tag{65}$$

where $\zeta(t)$ is as in (61). The other integrals can be estimated analogously.

The part of (64) generated by the integral (65) can be written in the form

$$\sum_{r,s=0}^{1} \iint_{I_r^1 \times I_s^2} f_\beta(y) D_y^{\alpha+\beta} \Delta_y^\mu G_{r,s}(x, y) \, dy_2 \, dy_1$$

where $I_0^j = (-\infty, x_j)$, $I_1^j = (x_j, \infty)$, $j = 1, 2$, and

$$G_{r,s}(x, y) = \int_{-|x-y|}^{|x-y|} \int_{|x-y| \geqslant |t_2| \geqslant |x|} F(x-y, {}_{r,s}\tilde{\zeta}(t))\, {}_r\tilde{\zeta}_1'(t_1)\, {}_s\tilde{\zeta}_2'(t_2) \, dt_2 \, dt_1$$

with

$${}_{r,s}\tilde{\zeta}(t) = ({}_r\tilde{\zeta}_1(t_1), {}_s\tilde{\zeta}_2(t_2)),$$

$${}_r\tilde{\zeta}_1(t_1) = \sigma^*(t_1) + (-1)^r i\tau(t_1),$$

$${}_s\tilde{\zeta}_2(t_2) = \sigma(t_2) + (-1)^s i\tau(t_2).$$

But, for $|\eta| \leqslant m \log|\zeta|$ and $|\zeta| \geqslant C_m$, the function $[\hat{S}(\zeta)\langle\zeta, \zeta\rangle^\mu]^{-1}$ is bounded together with all its derivatives, because of (51). Therefore, if $|x-y|$ is sufficiently large and $\geqslant |x|$, we have

$$|D_y^{\alpha+\beta}\,\Delta_y^{\mu}\,G_{r,s}(x,y)| \leqslant \exp\{-c|x|^p + a(2\mu + k + l + 3)|x-y|^p\},$$

$$r, s = 0, 1.$$

For a given $b > 0$ we now set $c = 2b$ and choose ρ (or equivalently l) so that $\rho > b$. Next we choose a so small that $a(2\mu + k + l + 3) < b/2^p$. Again m must be sufficiently large, because of (50). It follows that

$$\iint_{I_r^1 \times I_s^2} f_\beta(y) D_y^{\alpha+\beta}\,\Delta_y^{\mu}\,G_{r,s}(x,y)\,dy_2\,dy_1 = O(e^{-b|x|^p})$$

as $|x| \longrightarrow \infty$, which proves (63) and thus the lemma.

REMARK. Note that condition (51) is not inconsistent with the choice of the constants in the proof of Lemma 4, since ϵ can be chosen arbitrarily small.

8. **Sufficiency of condition (h_3).** In order to establish the sufficiency of (h_3) we need only to prove

THEOREM 10. *Let S be a distribution in $\mathcal{O}_c'(K_p' : K_p')$. If for every pair (b, k) there exists a (b, k)-parametrix for S, then S is hypoelliptic in K_p'.*

PROOF. Suppose that U is a solution in K_p' of the equation

$$S * U = V \tag{66}$$

where $V \in \mathcal{E}K_p'$. We have to prove that, in fact, $U \in \mathcal{E}K_p'$.

By Theorem 1, we can write $U = D^\beta f$ where f is a continuous function such that

$$f(x) = O(e^{b_1|x|^p}) \quad \text{as } |x| \longrightarrow \infty, \tag{67}$$

for some constant $b_1 > 0$.

On the other hand, since $V \in \mathcal{E}K_p'$, V is a C^∞-function such that

$$D^\alpha V(x) = O(e^{b_2|x|^p}) \quad \text{as } |x| \longrightarrow \infty, \tag{68}$$

for some $b_2 > 0$ and all α.

Suppose now that l is any given positive integer. By assumption there exists a (b, k)-parametrix P for S with $b = b_1 + b_2 + 1$ and $k = l + |\beta|$. Thus we have

$$S * P = \delta - W \tag{69}$$

where P and W satisfy the growth conditions in (p_1) and (p_2).

From (66) and (69) it now follows that

$$\begin{aligned} U = U * \delta &= U * (S * P) + U * W \\ &= (U * S) * P + U * W = V * P + U * W \end{aligned}$$

where the convolutions are well defined and the associativity is legitimate because of the rate of decrease of P and W.

But $V * P$ is in $\mathcal{E}K'_p$, since by (p_1),

$$D^\alpha(V * P) = \sum_{|\alpha'|\leq m} (D^{\alpha+\alpha'}V) * f_{\alpha'},$$

where

$$f_{\alpha'}(x) = O(e^{-b|x|^p}) \quad \text{as } |x| \longrightarrow \infty,$$

so that $V * P$ is a C^∞-function and, by (68),

$$D^\alpha(V * P)(x) = O(e^{b_2|x|^p}) \quad \text{as } |x| \longrightarrow \infty$$

for all α.

Also $U * W = f * D^\beta W$, which shows that $U * W$ is a function in C^l and, by (67),

$$D^\alpha(U * W)(x) = O(e^{b_1|x|^p}) \quad \text{as } |x| \longrightarrow \infty$$

for $|\alpha| \leq l$.

Consequently U is a function in C^l and

$$D^\alpha U(x) = e^{(b_1+b_2)|x|^p} \quad \text{as } |x| \longrightarrow \infty$$

for $|\alpha| \leq l$. But l was arbitrary and therefore U must be in $\mathcal{E}K'_p$.

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